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# SCREW DISLOCATION IN A NON-LINEAR ELASTIC CONTINUUM

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**Abstract**—The problem of a screw dislocation in a "physically" non-linear elastic continuum is considered. It is shown that, for a wide class of non-linear elastic materials, the solution is given by the same displacement field as in the case of linear elasticity.

The problem of a screw dislocation in a "physically" non-linear infinite isotropic elastic continuum is considered here; strains and rotations are assumed to be small.

The displacement field associated with a screw dislocation is discontinuous; in the case of linear elasticity it is given by the formulae (see, e.g. [1]):

$$u_x = 0,\ u_y = 0,\ u_z = -\frac{b}{2\pi}\frac{y}{x} \tag{1}$$

so that the strain field is

$$\epsilon_{xz} = \frac{b}{4\pi}\frac{y}{x^2+y^2},\ \epsilon_{yz} = -\frac{b}{4\pi}\frac{x}{x^2+y^2},\ \text{other } \epsilon_{ij} = 0 \tag{2}$$

where $z$ is the dislocation line, $y$ is the axis perpendicular to the slip plane and $b$ is the magnitude of the Burgers vector.

It is shown here that for a wide class of nonlinear elastic materials the displacement field due to a screw dislocation is given by the same formulae (1) although the stress field will differ from the linear elastic one.

Assume, first, that the constitutive law has the form of a nonlinear relation between the stress and strain deviators $\mathbf{D}_\sigma$ and $\mathbf{D}_\epsilon$ similar to the one used in the deformational theory of plasticity:

$$\mathbf{D}_\sigma = 2g(\Gamma)\mathbf{D}_\epsilon \tag{3}$$

where $g(\Gamma)$ is a function of the shearing strain intensity $\Gamma = \{2[\epsilon_{ij}\epsilon_{ij} - (\epsilon_{kk})^2]\}^{1/2}$. (For a linear elastic body $g(\Gamma) = \text{const} = G$ (shear modulus)). Then the stress field corresponding to strains (2) is

$$\sigma_{xz} = 2g(\Gamma)\epsilon_{xz},\ \sigma_{yz} = 2g(\Gamma)\epsilon_{yz},\ \text{other } \sigma_{ij} = 0. \tag{4}$$

The conditions of equilibrium, $\partial\sigma_{ij}/\partial x_i = 0$, take the form

$$g'(\Gamma)\left(\frac{\partial\Gamma}{\partial x}\epsilon_{xz} + \frac{\partial\Gamma}{\partial y}\epsilon_{yz}\right) + g(\Gamma)\left(\frac{\partial\epsilon_{xz}}{\partial x} + \frac{\partial\epsilon_{yz}}{\partial y}\right) = 0. \tag{5}$$

The shearing strain intensity $\Gamma$ is $[2(\epsilon_{xz}^2 + \epsilon_{yz}^2)]^{1/2}$, and calculations using (2) show that the first parenthesis in (5) vanishes. The second parenthesis also vanishes since the field (2) is a solution of a linear elastic problem. Thus the displacement and strain fields (1) and (2) provide a solution for the considered nonlinear elastic body, with the stresses given by the formulae (4). It should be emphasized that $g(\Gamma) = g((b/2\pi)(1/\sqrt{x^2+y^2}))$ is an arbitrary differentiable function of its argument.

This result is valid not only for the constitutive law (3) but for a wider class of nonlinear materials. Indeed since the first $I_1 = \epsilon_{kk}$ and the third $I_3 = \text{Det}|\epsilon_{ij}|$ invariants of the strain field associated with a screw dislocation are identically equal to zero, function $g$ can be taken as an arbitrary (differentiable) function of all three invariants of the strain tensor $\boldsymbol{\epsilon}$, i.e. an arbitrary scalar function of $\boldsymbol{\epsilon}$.

To obtain the general form of stress-strain relations for which the given solution is valid, we use the Hamilton–Cayley representation of an isotropic tensor function:

$$\boldsymbol{\sigma} = f(I_1, I_2, I_3)\mathbf{I} + 2g(I_1, I_2, I_3)\boldsymbol{\epsilon} + h(I_1, I_2, I_3)\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon} \tag{6}$$

($\mathbf{I}$ is a unit tensor, $I_1$, $I_2$, $I_3$ are the invariants of $\boldsymbol{\epsilon}$. For a linear elastic body $f = \lambda I_1$, $g = G$, $h = 0$). If the body considered is elastic but not necessarily "hyperelastic" (i.e. the requirement that the strain energy function exists is not imposed) then the class of constitutive laws for which the obtained result is valid is determined by the following restriction: functions $f$ and $h$ must be differentiable and vanish when either $I_1$ and $I_3$ become equal to zero, otherwise they can be arbitrary; $g$ is an arbitrary differentiable function.

To ensure the existence of the strain energy function $\Phi = \Phi(I_1, I_2, I_3)$, some additional restrictions must be imposed on $f$, $g$ and $h$. Since differentation of the function of invariants $\Phi(I_1, I_2, I_3)$ with respect to symmetric tensor $\boldsymbol{\epsilon}$ gives:

$$\frac{\partial\Phi}{\partial\boldsymbol{\epsilon}} = \left(\frac{\partial\Phi}{\partial I_1} + I_1\frac{\partial\Phi}{\partial I_2} + I_2\frac{\partial\Phi}{\partial I_3}\right)\mathbf{I} - \left(\frac{\partial\Phi}{\partial I_2} + I_1\frac{\partial\Phi}{\partial I_3}\right)\boldsymbol{\epsilon} + \frac{\partial\Phi}{\partial I_3}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}$$

the coefficients $f$, $g$ and $h$ in representation (6) must be related to derivatives of $\Phi(I_1, I_2, I_3)$ as follows:

$$f = \frac{\partial\Phi}{\partial I_1} + I_1\frac{\partial\Phi}{\partial I_2} + I_2\frac{\partial\Phi}{\partial I_3},\ 2g = -\left(\frac{\partial\Phi}{\partial I_2} + I_1\frac{\partial\Phi}{\partial I_3}\right),\ h = \frac{\partial\Phi}{\partial I_3}. \tag{7}$$

Elimination of $\Phi(I_1, I_2, I_3)$ from (7) yields the additional restrictions imposed on $f$, $g$ and $h$ by existence of the strain energy function ("integrability conditions"):

$$\left.\begin{aligned}
&\left(\frac{\partial}{\partial I_1} + I_1\frac{\partial}{\partial I_2}\right)(2g + I_1 h) + \frac{\partial}{\partial I_2}(f - I_2 h) = 0\\
&\frac{\partial h}{\partial I_2} + \frac{\partial}{\partial I_3}(2g + I_1 h) = 0\\
&\frac{\partial h}{\partial I_1} - I_1\frac{\partial}{\partial I_3}(2g + I_1 h) - \frac{\partial}{\partial I_3}(f - I_2 h) = 0
\end{aligned}\right\}. \tag{8}$$

These general restrictions can, however, be relaxed considerably in the case of screw dislocations: it is sufficient if relations (8) are satisfied only for $I_1 = I_3 = 0$. Since in this case, as has been found above, functions $f$ and $h$ must vanish the additional conditions take the form:

$$\underline{\text{at } I_1 = I_3 = 0}\qquad \left.\begin{aligned}
&2\frac{\partial g}{\partial I_1} + \frac{\partial f}{\partial I_2} = 0\\
&\frac{\partial h}{\partial I_2} + 2\frac{\partial g}{\partial I_3} = 0\\
&\frac{\partial h}{\partial I_1} - \frac{\partial f}{\partial I_3} - I_2\frac{\partial h}{\partial I_3} = 0
\end{aligned}\right\} \tag{9}$$

In particular, relations (9) do not impose any restrictions on function $g$ if the latter is chosen as a function of $\Gamma$ only and $h$ is taken to be identically zero (as in (4)); also, no restrictions are imposed on $f$ if it is chosen as a function of $I_1$ only.

## REFERENCE

1. A. H. Cottrell, *Dislocations and Plastic Flow in Crystals*. Clarendon Press, Oxford (1953).